Basics of Neural Networks and Convolutional Neural Networks

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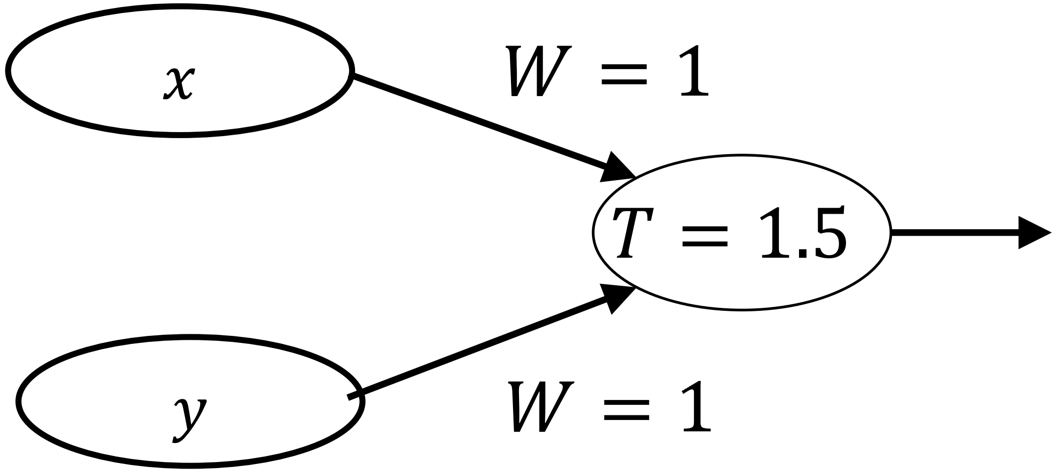
Perceptron



$$\text{output} = \begin{cases} 1 & \text{if } \displaystyle\sum_{i=0}^{n} W_i x_i > 0 \\ -1 & \text{otherwise} \end{cases}$$

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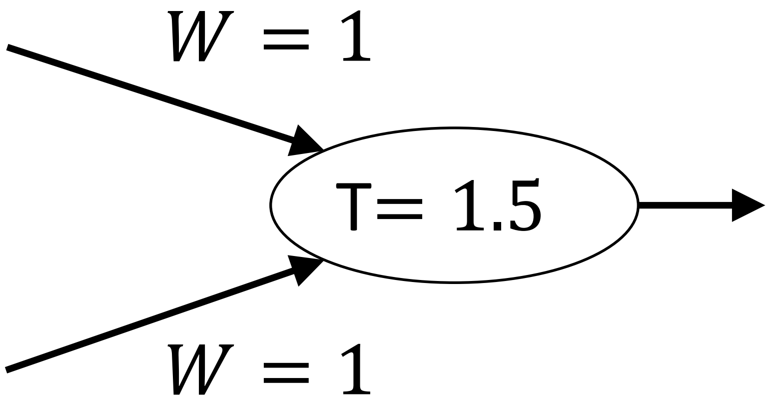

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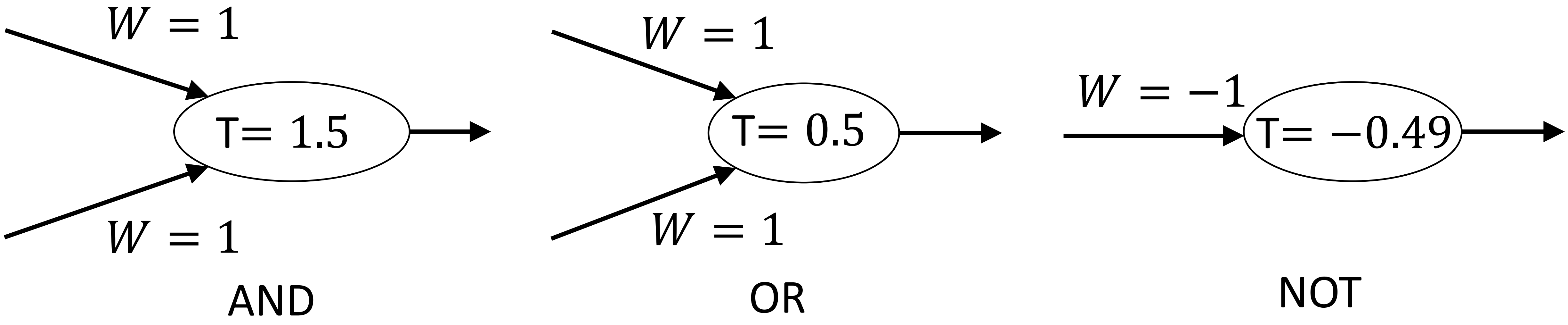
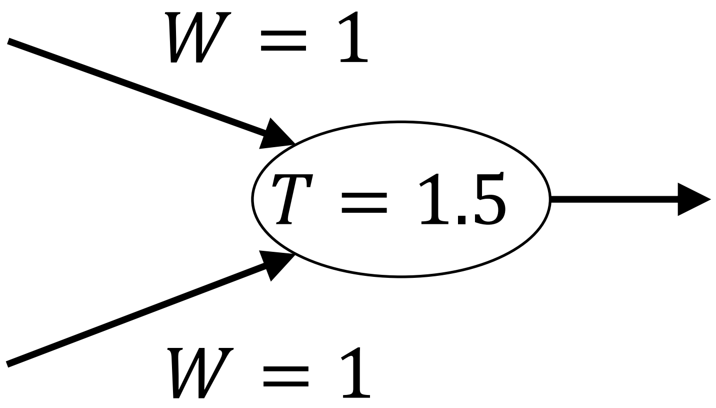
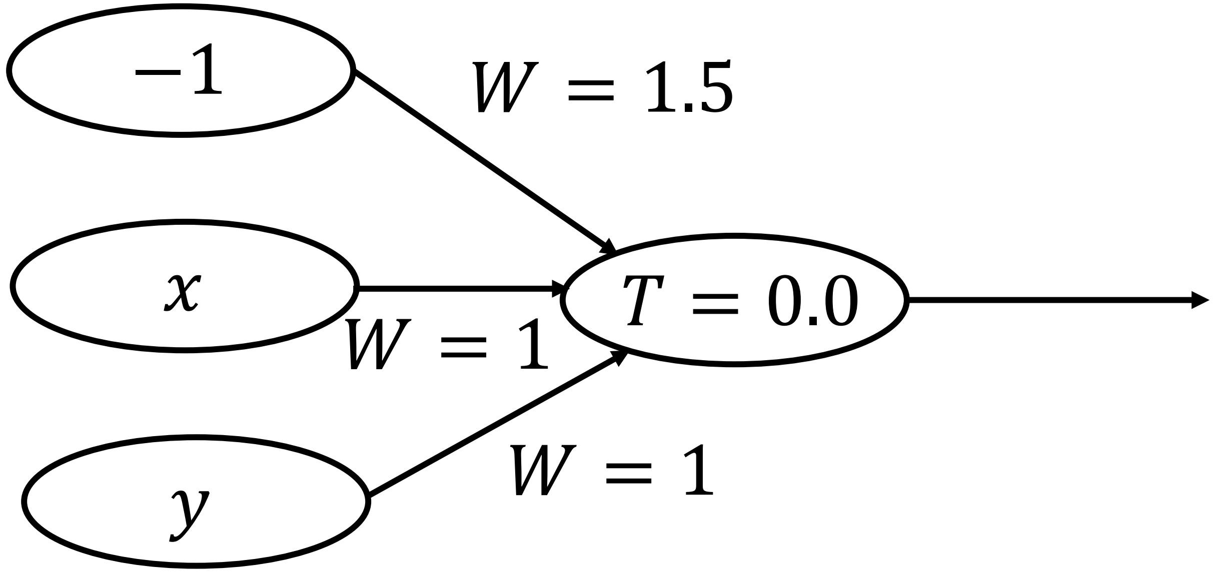
Perceptron



Perceptron: Linear Threshold Unit





$$output = \begin{cases} 1 & if \sum_{i=0}^{n} W_i x_i \;\; > 0 \\ -1 & otherwise \end{cases}$$

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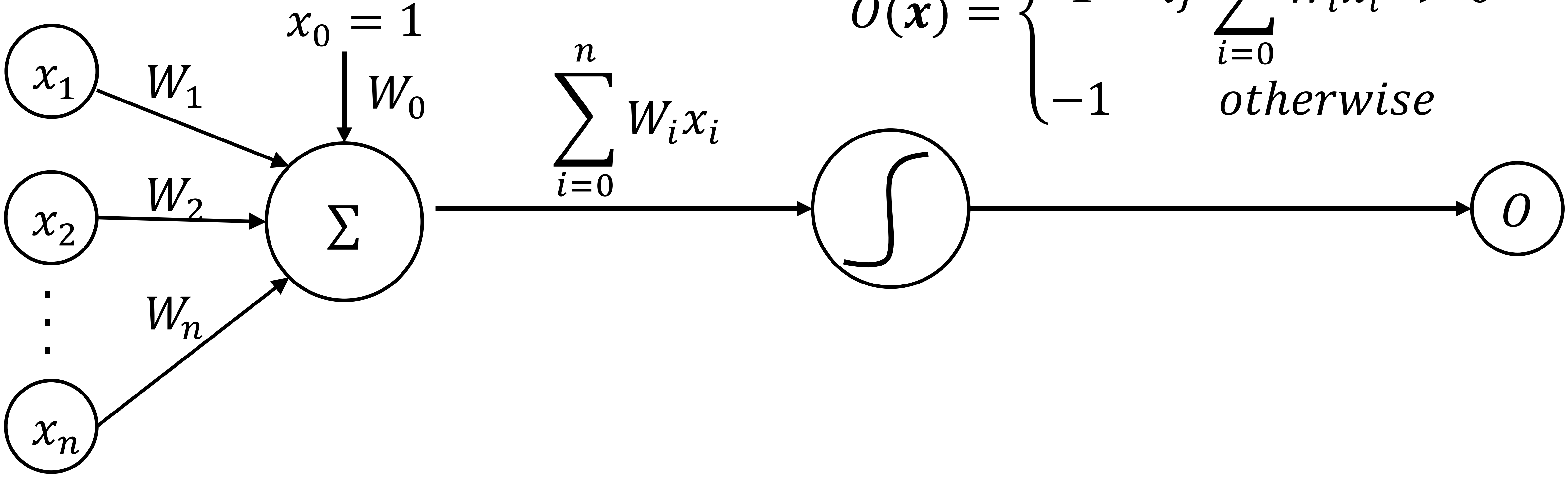
$$O(\boldsymbol{x}) = \begin{cases} 1 & if \sum_{i=0}^{n} W_i x_i > 0 \\ -1 & otherwise \end{cases}$$



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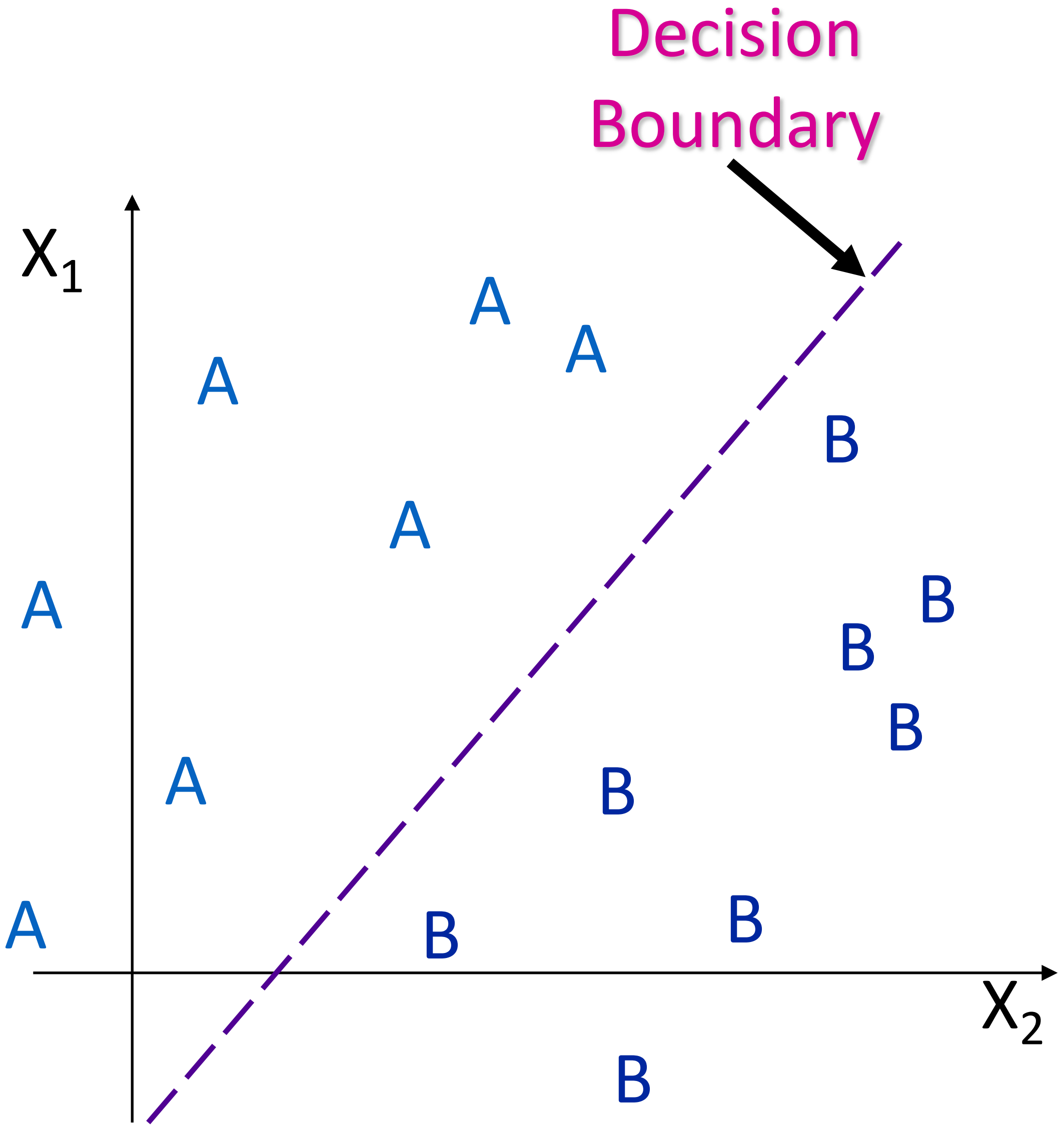
Decision boundaries

- In simple cases, divide feature space by drawing a hyperplane across it.
- Known as a decision boundary.
- Discriminant function: returns different values on opposite sides. (straight line)
- Problems which can be thus classified are linearly separable.



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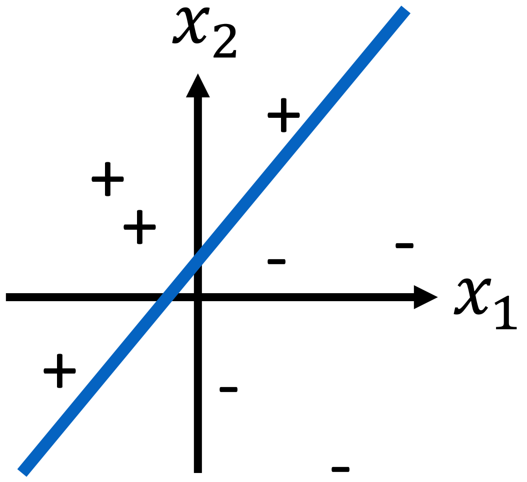
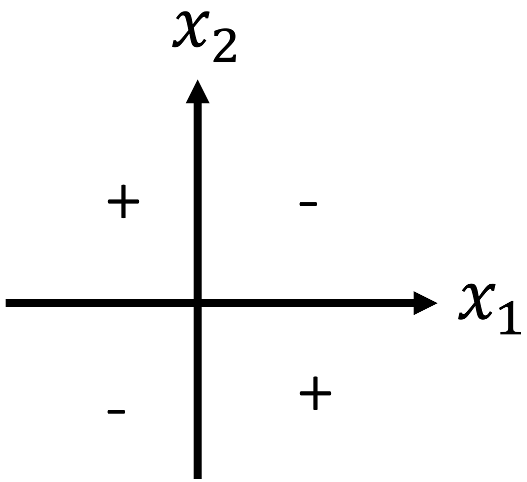
Decision Surface of a Perceptron

- Perceptron is able to represent some useful functions
- $AND(x_1, x_2)$ choose weights $w_0 = -1.5, w_1 = 1, w_2 = 1$
- But functions that are not linearly separable (e.g. XOR) are not representable



Linearly separable



Non-Linearly separable

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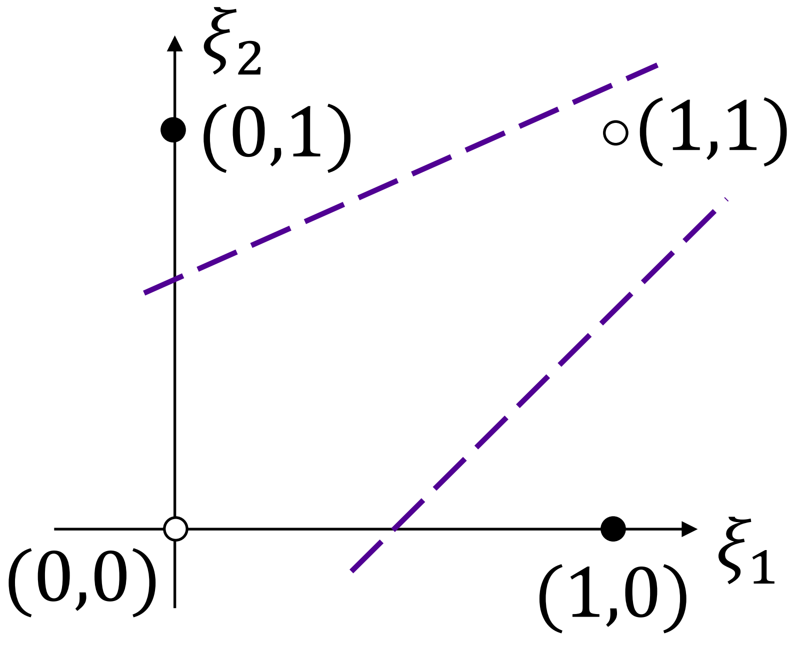
Solution in 1980s: Multilayer Perceptrons

- Removes many limitations of single-layer networks
  - Can solve XOR

- Exercise: Draw a two-layer perceptron that computes the XOR function
  - 2 binary inputs ξ_1 and ξ_2
  - 1 binary output
  - One “hidden” layer
  - Find the appropriate weights and threshold



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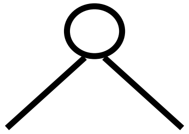
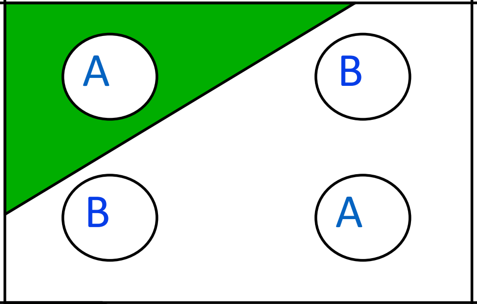
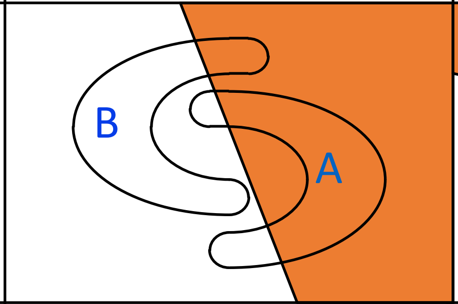
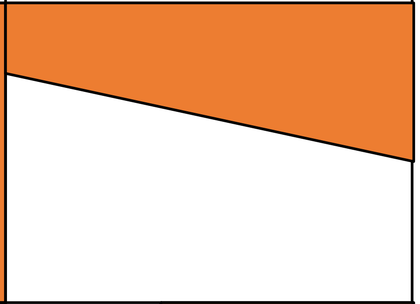
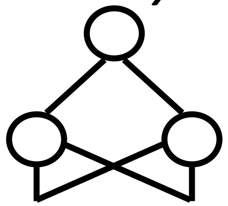
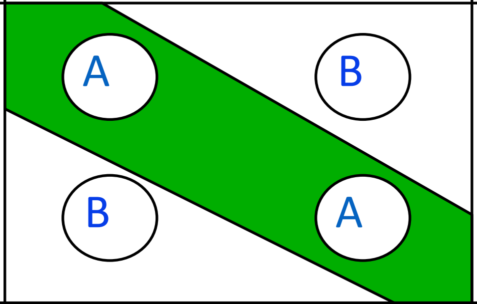
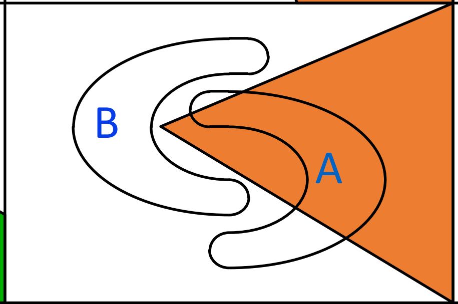
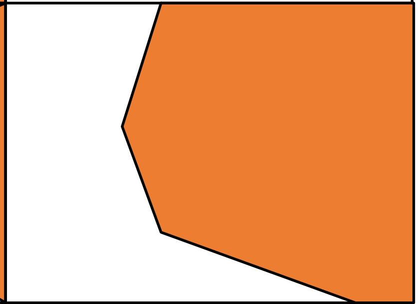
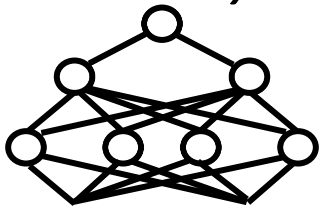
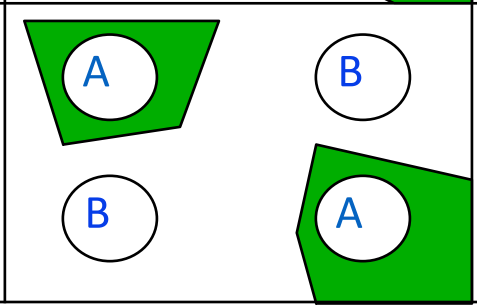
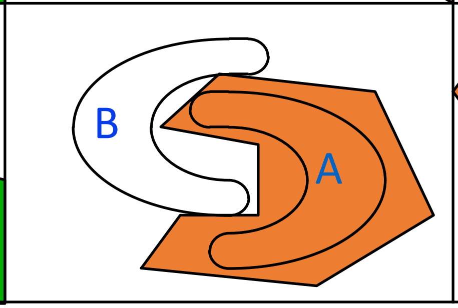
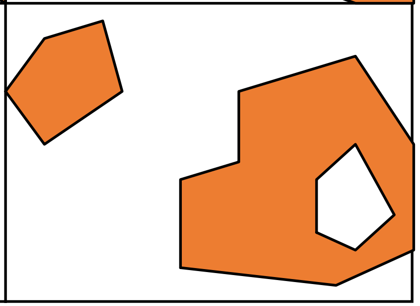
Different Non-Linearly Separable Problems

Structure	*Types of Decision Regions*	*Exclusive-OR Problem*	*Classes with Meshed regions*	*Most General Region Shapes*
Single-Layer 	*Half Plane Bounded By Hyperplane*			
Two-Layer 	*Convex Open Or Closed Regions*			
Three-Layer 	Arbitrary (Complexity Limited by No. of Nodes)			

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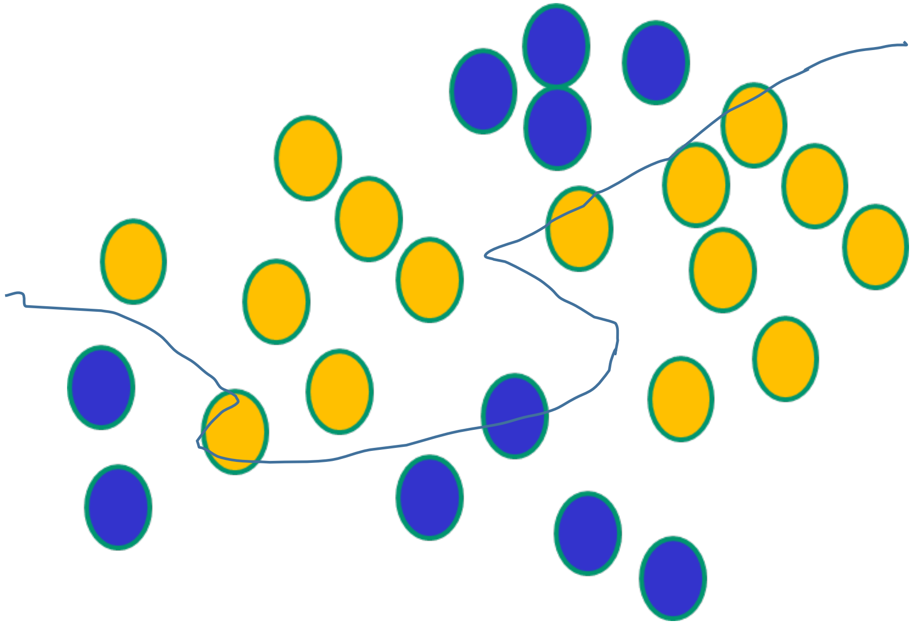
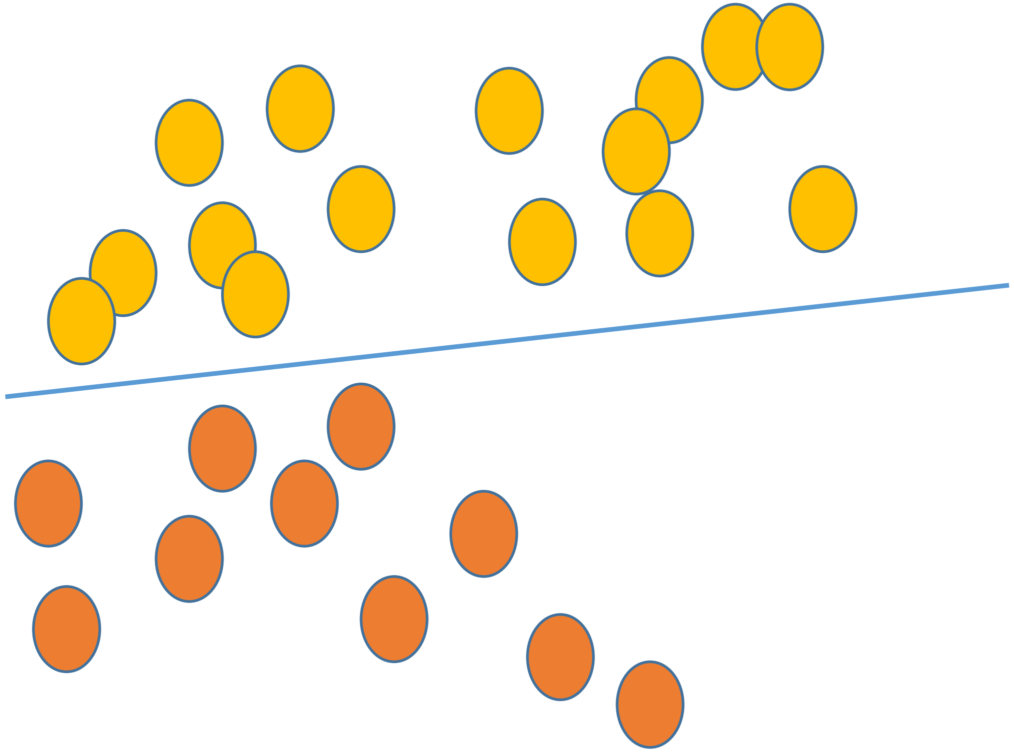
Neural Networks: Two Perspectives

Use nonlinear $f(x)$ to draw complex boundaries, but keep the data unchanged



Find linear boundaries only, but transform the data first in a way that makes linear boundaries OK



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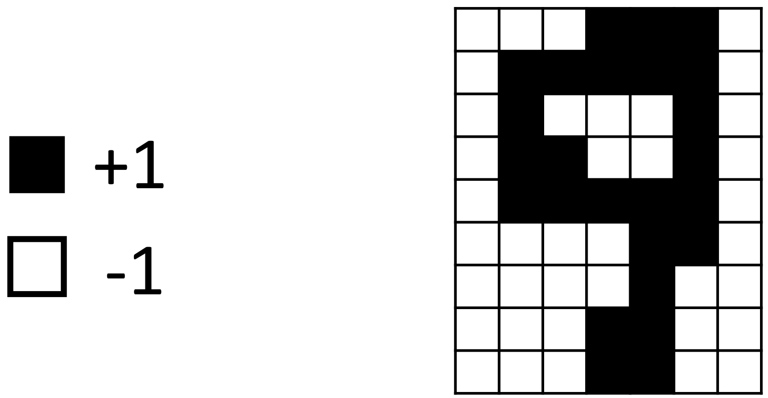
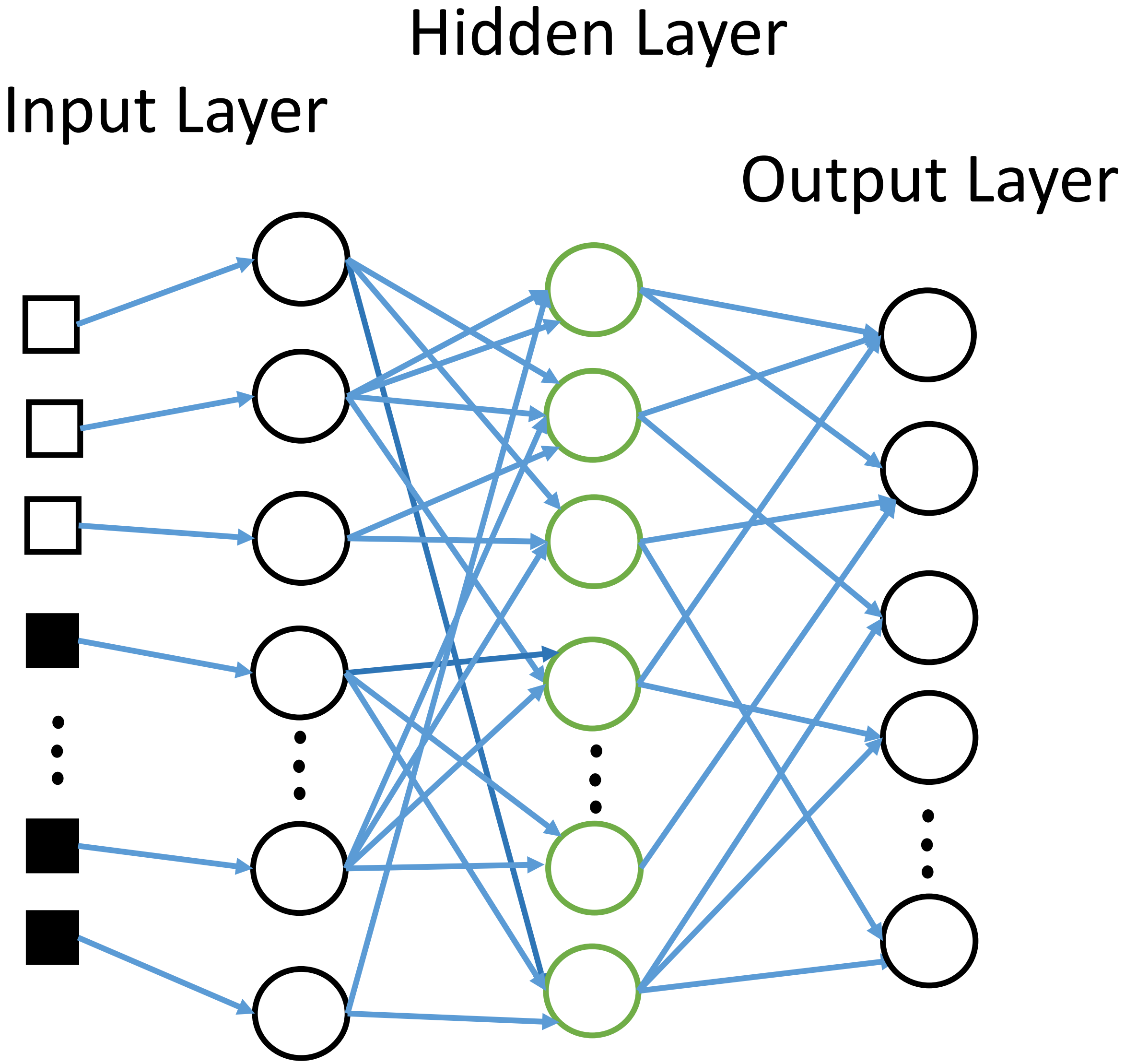
Neural Networks as Learning Representations

Handwritten images





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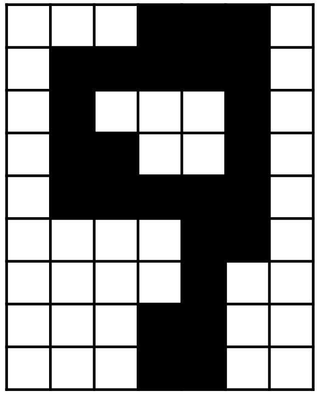
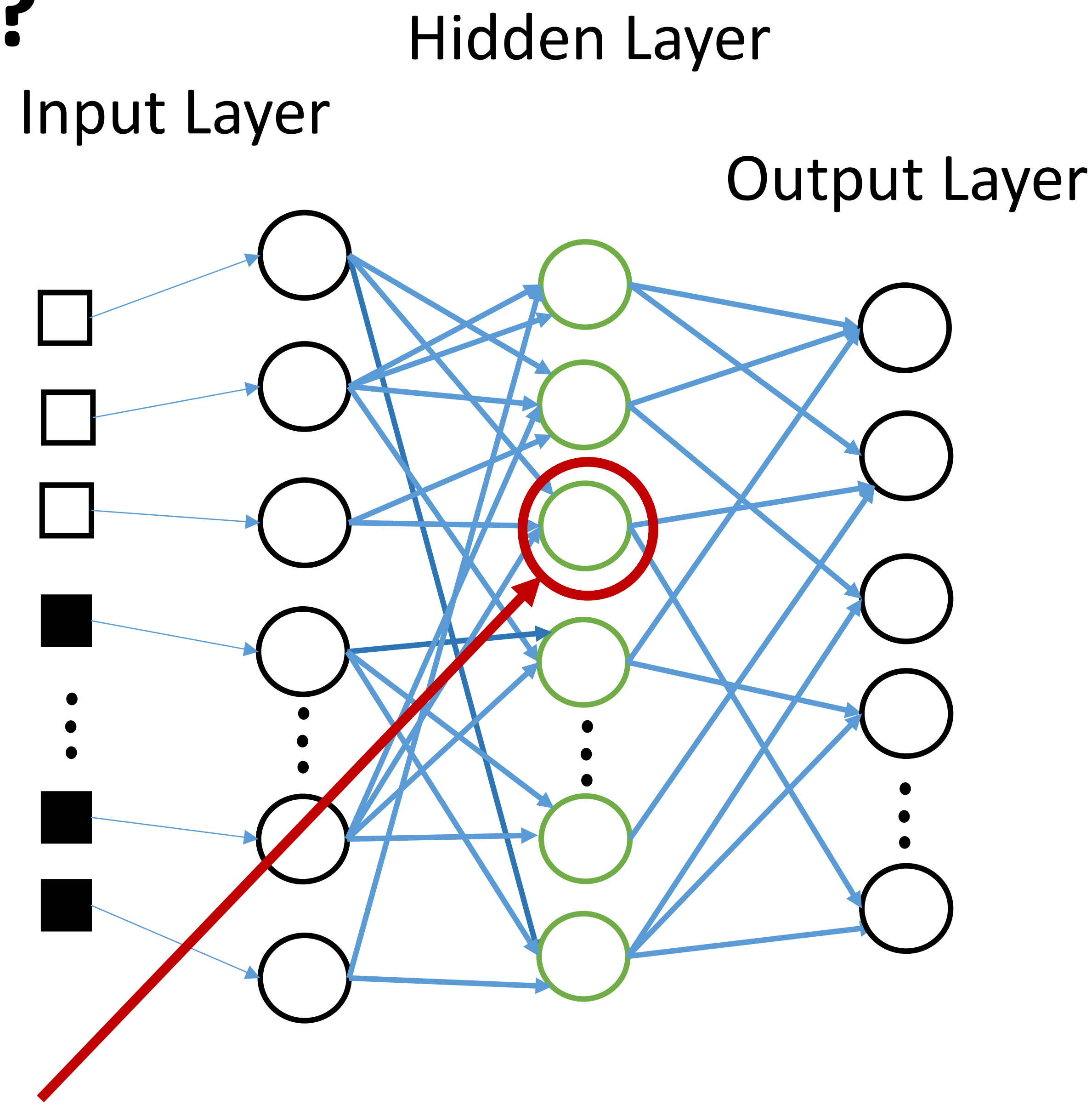
What is this unit doing?

Handwritten images





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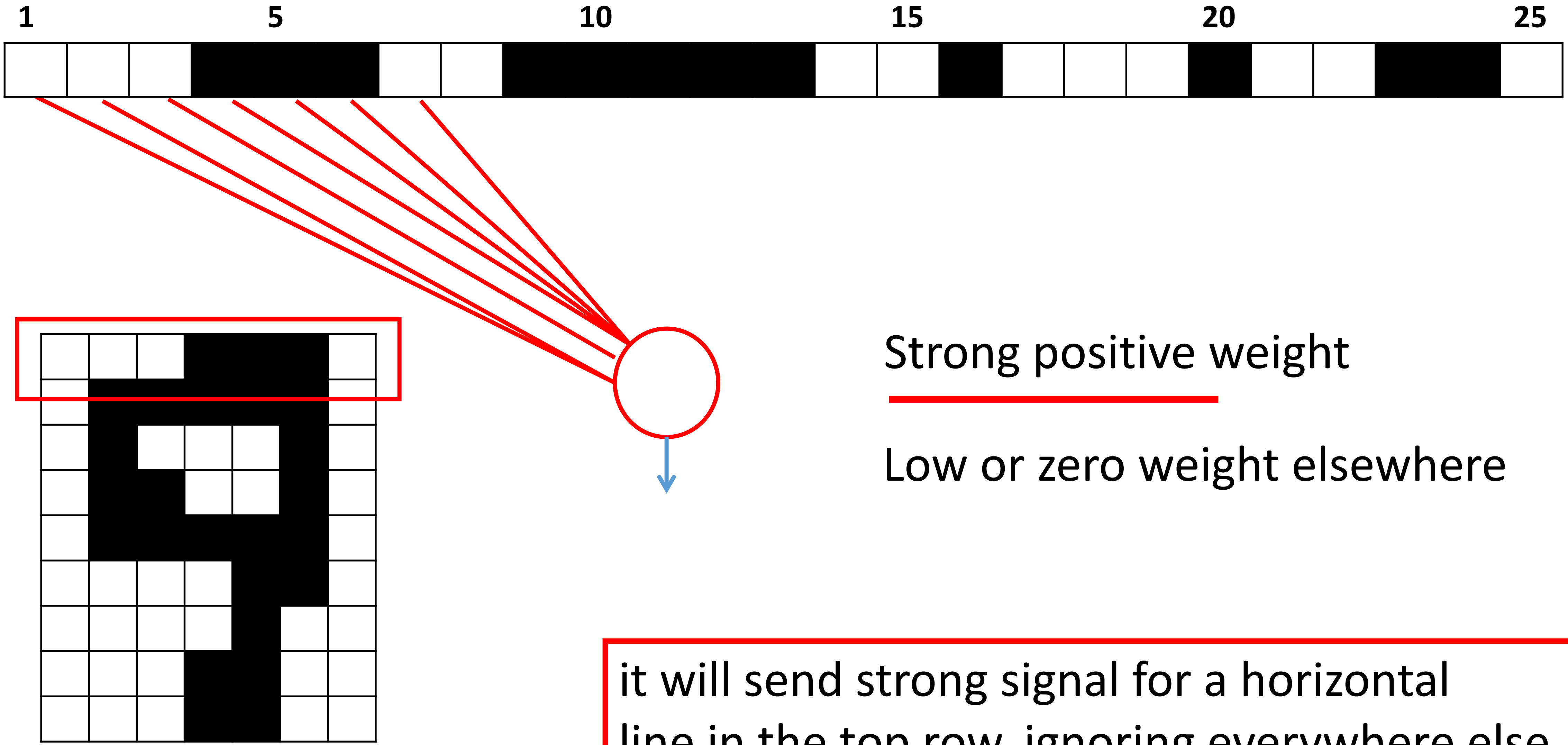
What does this unit detect?



Strong positive weight

Low or zero weight elsewhere

it will send strong signal for a horizontal line in the top row, ignoring everywhere else

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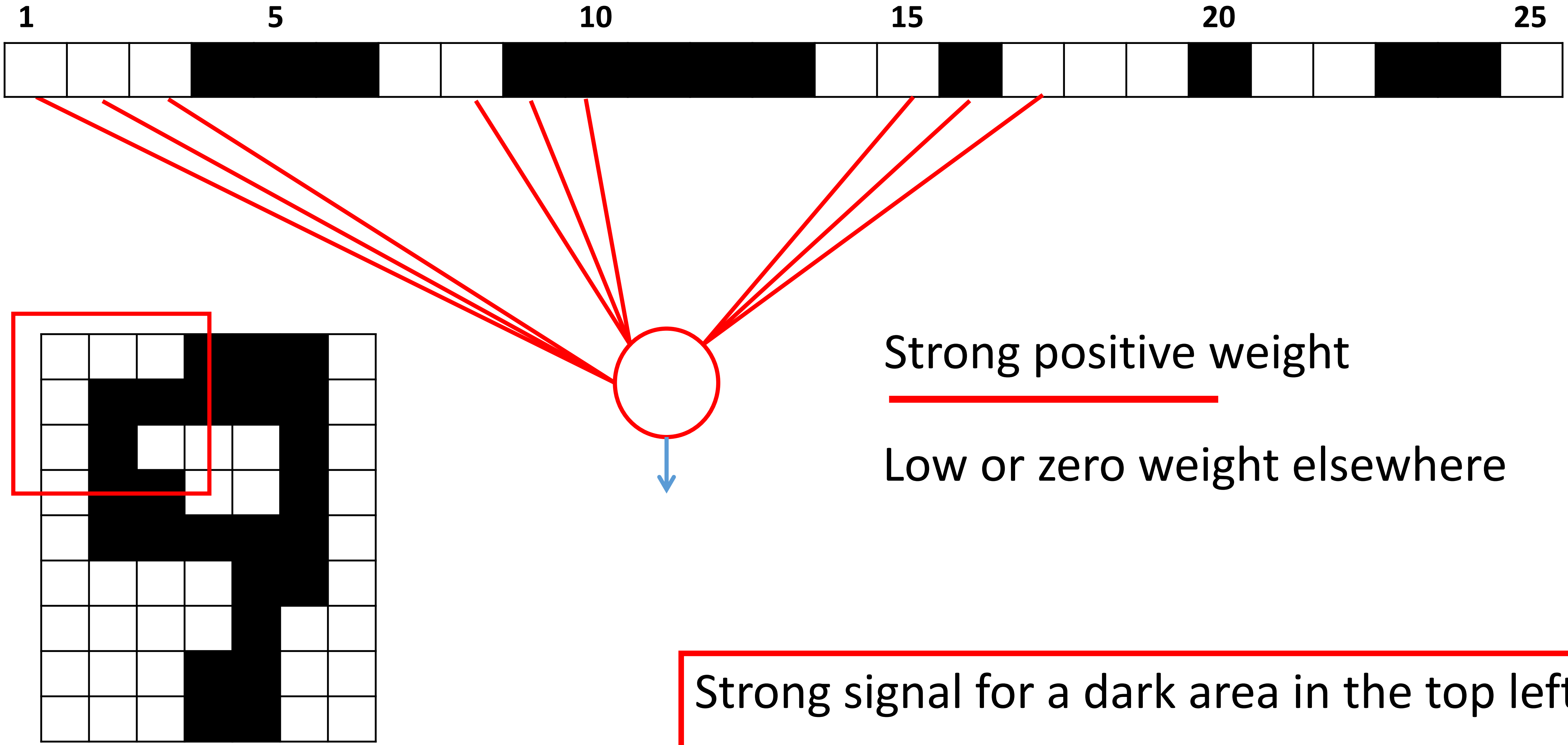
What does this unit detect?



Strong positive weight

Low or zero weight elsewhere

Strong signal for a dark area in the top left corner

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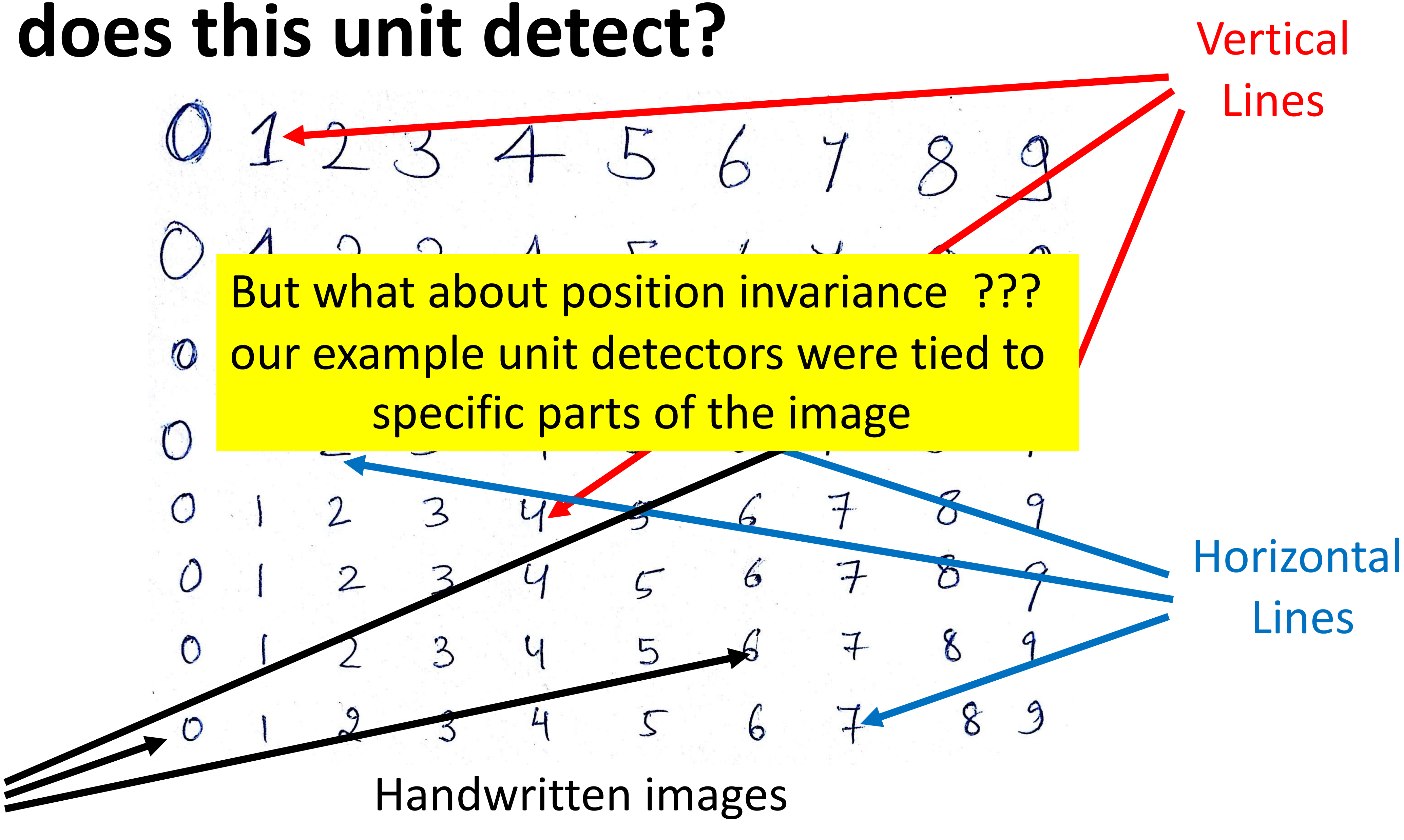
What does this unit detect?



But what about position invariance ???
our example unit detectors were tied to specific parts of the image

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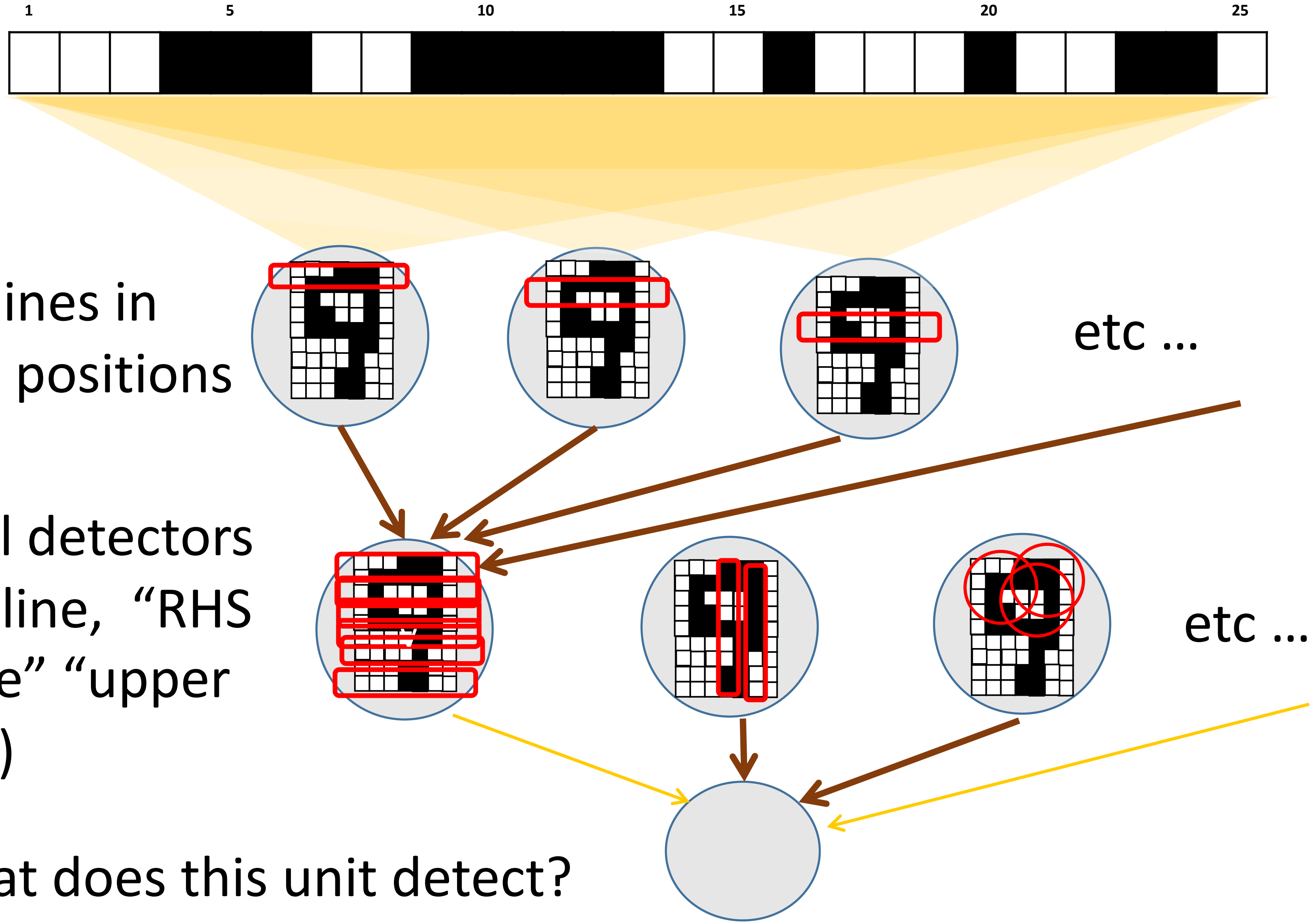
Successive layers can learn higher representations



detect lines in specific positions

Higher level detectors (horizontal line, "RHS vertical lune" "upper loop", etc...)

What does this unit detect?

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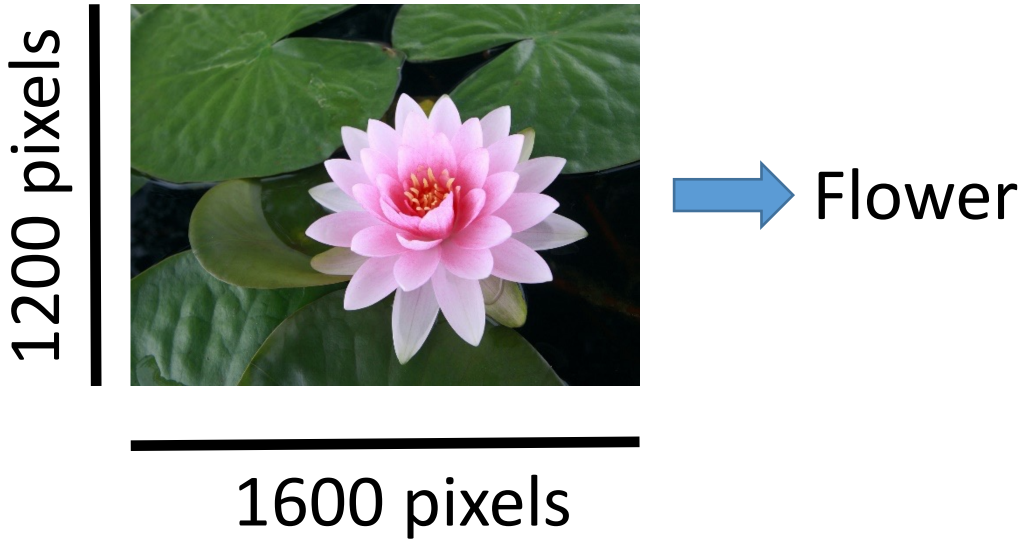
Scaling-up

- Challenges of practical image classification
  - Must deal with very high-dimensional inputs
  - 1600×1200 pixels = 1.9m inputs, or 3×1.9m if RGB pixels
  - Can we exploit the 2D topology of pixels (or 3D for video data)?
  - Can we build invariance to certain variations we can expect
    - Translations, illumination, etc.



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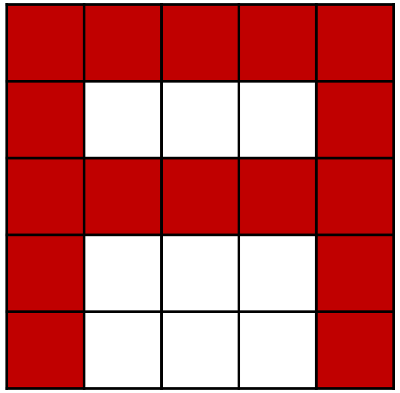
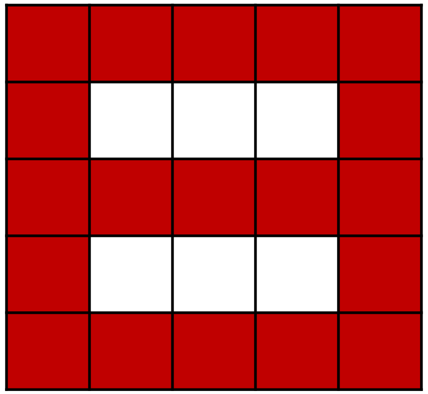
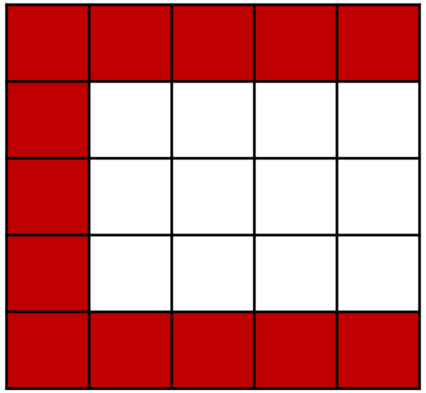
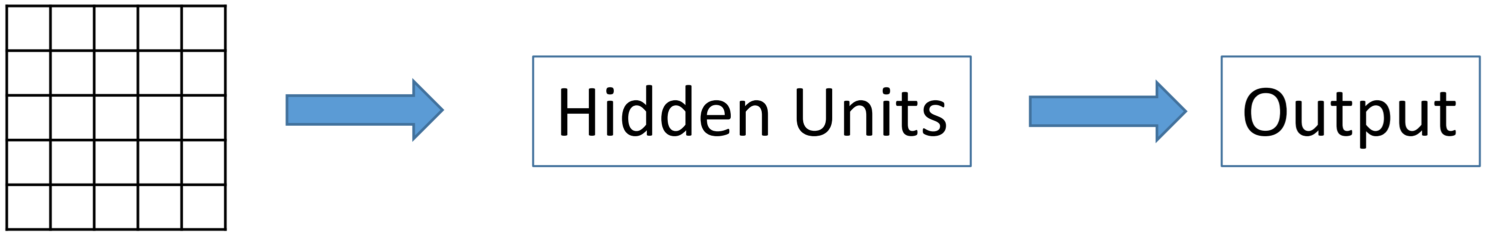
Motivation: Recognizing an Image

- Input is 5x5 pixel array

- Simple back propagation net



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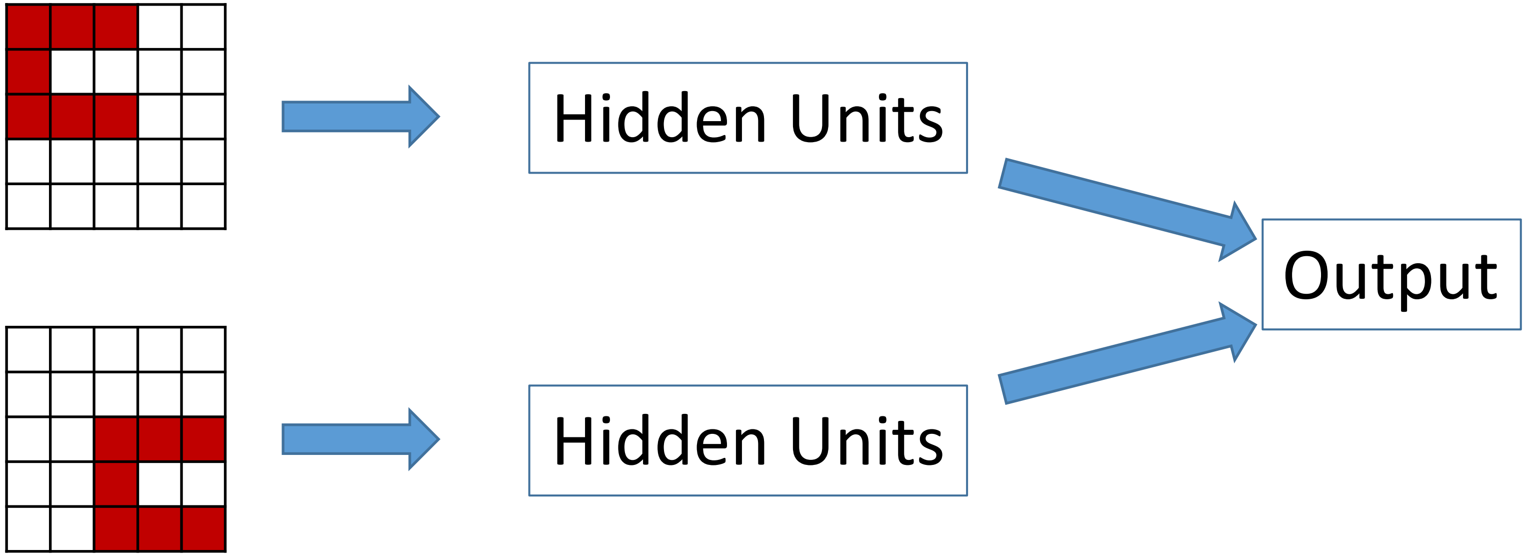
Recognizing an Image with unknown location

- Object can appear either in the top image or in the bottom image location
- Output indicates presence of object regardless of position
- What do we know about the weights?



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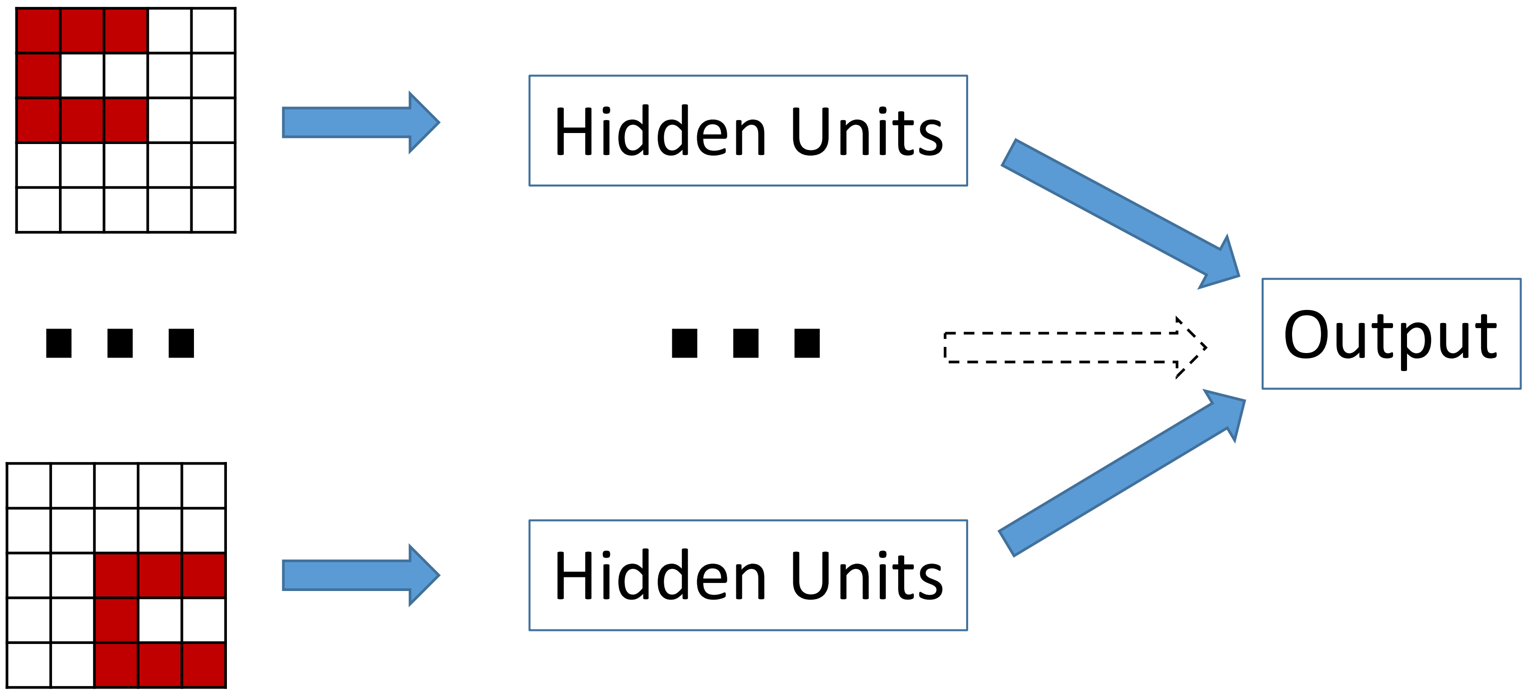
Recognizing an Image with any location

- Each possible location the object can appear in has its own set of hidden units
- Each set detects the same features except in a different location
- Locations can overlap



Key Ideas for a Usable ML Model for Images

Images:

1. Local Structure
2. Reusable Composition
3. Hierarchical

Must exploit each of them

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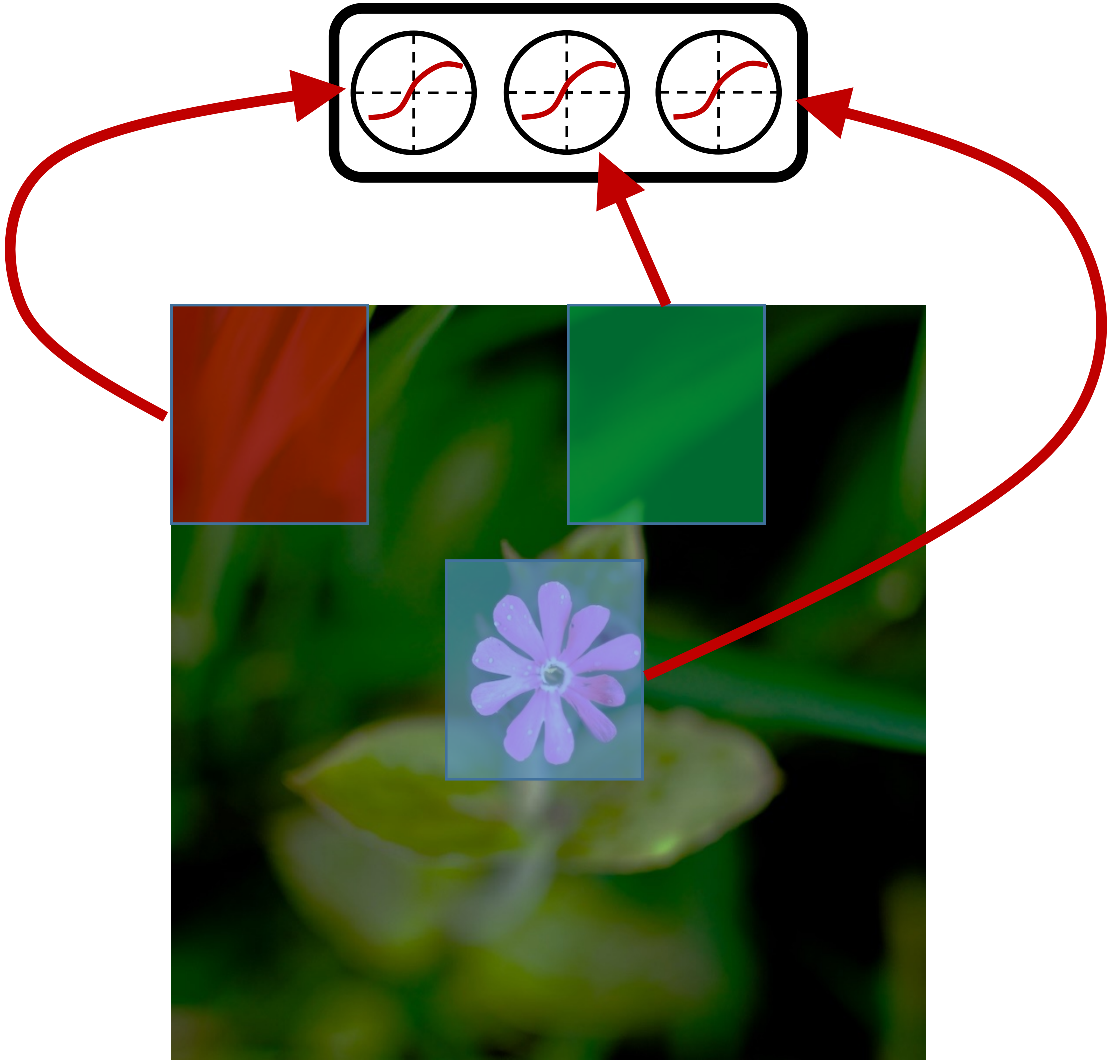
CNN: Local Connectivity



CNN: Local Connectivity

- Each hidden unit is connected only to a sub-region (patch) of the input image
- It is connected to all channels
  - 1 if greyscale image
  - 3 (R, G, B) for color image

Solves the following problems:

- Fully connected hidden layer would have an unmanageable number of parameters
- Computing the linear activations of the hidden units would be very expensive

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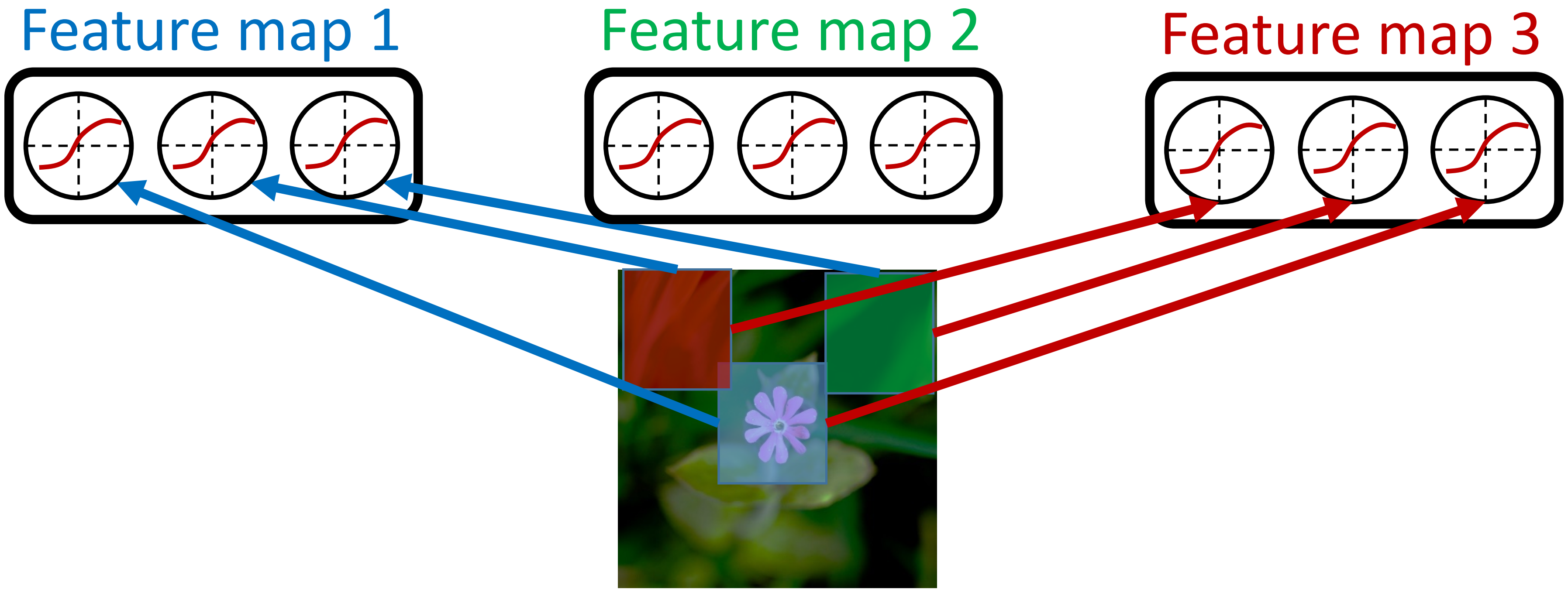
CNN: Parameter Sharing

- Units organized into the same “feature map/template” share parameters
- In this way all neurons detect the same feature/template at different positions in the input image
- Hidden units within a feature map cover different positions in the image



CNN: Parameter Sharing

How does it help?

- Reduces even more number of parameters (compared to local connectivity)
- Will extract the same features at every position
    - features are “equi-variant”

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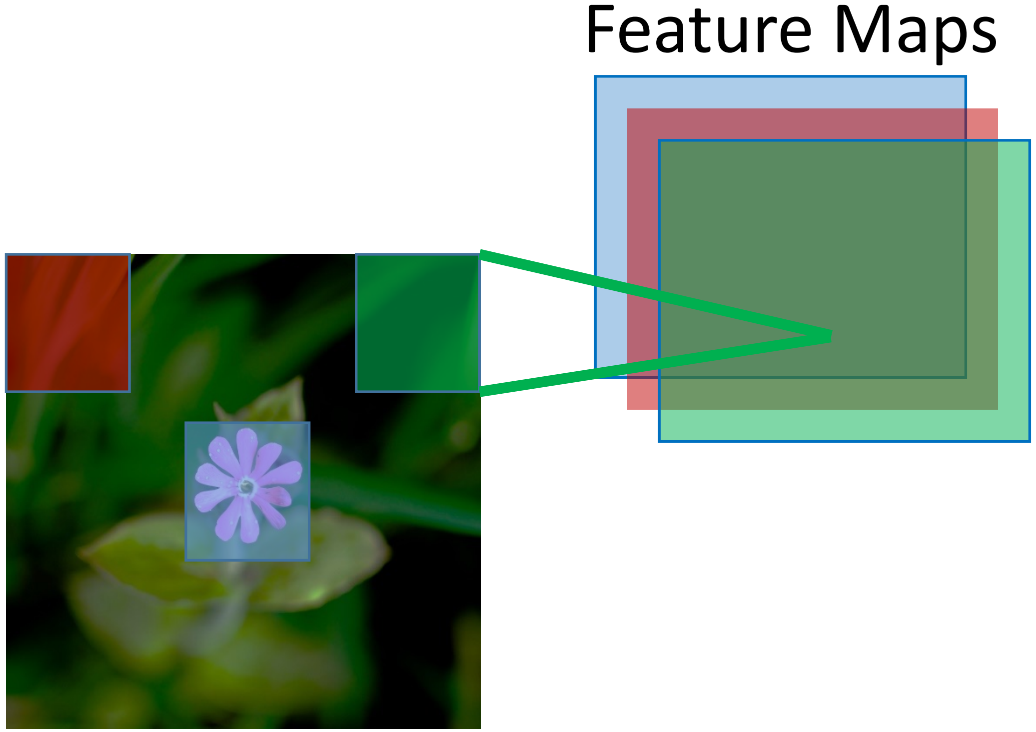
CNN: Parameter Sharing

- Each feature map forms a 2D grid of features
- Can be computed with a discrete convolution ($*$) of a kernel matrix k_{ij} which is the hidden weights matrix W_{ij} with its rows and columns flipped:

$$y_j = \tanh \sum_i k_{ij} x_i$$

- x_i is the i^{th} channel of input
- k_{ij} is the convolution kernel
- y_j is the hidden layer



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CNN: Parameter Sharing

- **Stride** (typically denoted by **s** in CNNs) decides the spacing between overlapping convolutions

- Helps in reducing parameters further

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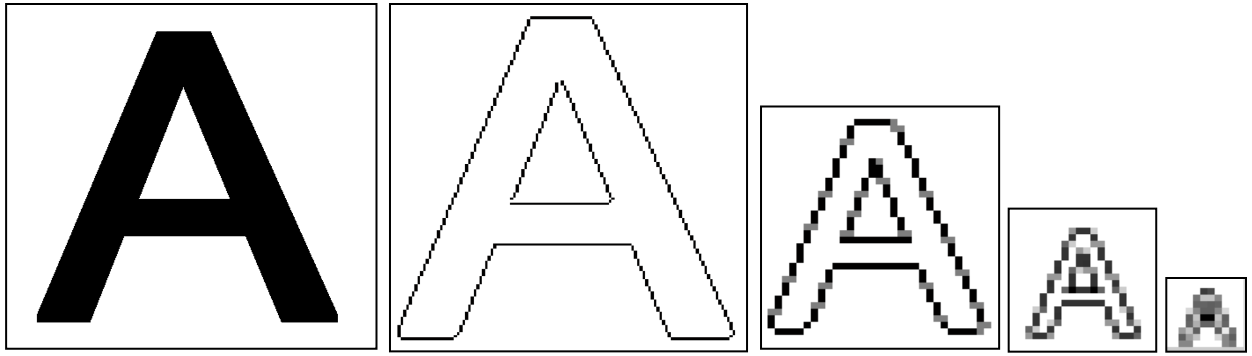
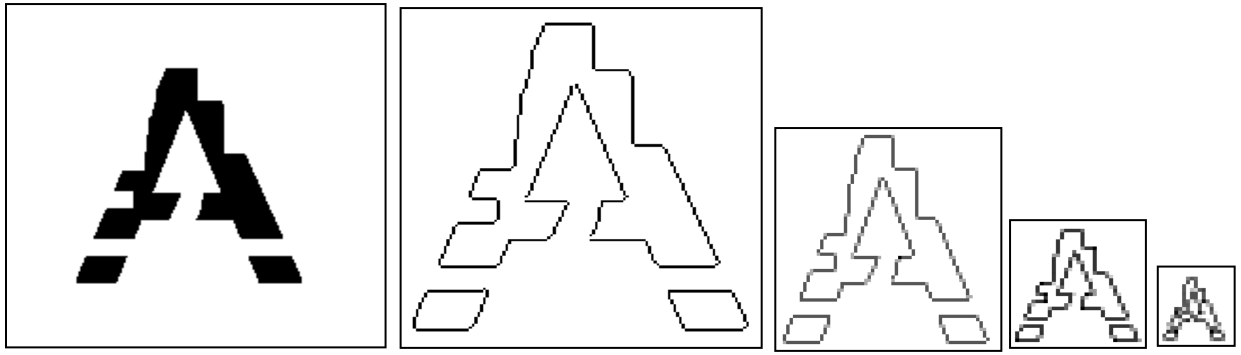
Pooling or Subsampling Layer

- The pooling layers reduce the spatial resolution of each feature map. Helps to collate the features in a region

- By reducing the spatial resolution of the feature map, a certain degree of noise, shift and distortion invariance is also achieved.





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Jargon

- Convolutional Neural Networks
  - also called CNNs, Conv Nets etc.
- Each hidden unit channel
  - also called **map, feature, feature type, dimension**
- Weights for each channel
  - also called **kernels**
- Input patch to a hidden unit at (x,y)
  - also called **receptive field**

Typical CNN

- Alternates convolutional and pooling layers

- Output layer is a regular, fully connected layer with softmax non-linearity
  - Output provides an estimate of the conditional probability of each class

- The network is trained by stochastic gradient descent
  - Backpropagation is used similarly as in a fully connected network
  - We have seen how to pass gradients through element-wise activation function
  - We also need to pass gradients through the convolution operation and the pooling operation

Backpropagation

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Backpropagation in General Cases

1. Decompose operations in layers of a neural network into function elements whose derivatives w.r.t. inputs are known by symbolic computation.

$$h_\theta(x) = \left(f^{(l_{max})} \circ \cdots \circ f^{(l)}_{\theta^{(l)}} \circ \cdots \circ f^{(2)}_{\theta^{(2)}} \circ f^{(1)}\right)(x)$$

where $f^{(1)} = x$ $\quad$ $f^{(l_{max})} = h_\theta$

and $\forall l: \dfrac{\partial f^{(l)}}{\partial f^{(l-1)}}$ is known

Backpropagation in General Cases

2. Backpropagate error signals corresponding to a differentiable cost function by numerical computation (Starting from cost function, plug in error signals backward).

$$\delta^{(l)} = \frac{\partial}{\partial f^{(l)}} J(\theta; x, y) = \frac{\partial J}{\partial f^{(l+1)}} \frac{\partial f^{(l+1)}}{\partial f^{(l)}} = \delta^{(l+1)} \frac{\partial f^{(l+1)}}{\partial f^{(l)}}$$

where $\frac{\partial J}{\partial f^{(l_{max})}}$ is known

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Backpropagation in General Cases

3. Use back-propagated error signals to compute gradients w.r.t. parameters only for the function elements with parameters where their derivatives w.r.t. parameters are known by symbolic computation.

$$\nabla_{\theta^{(l)}} J(\theta; x, y) = \frac{\partial}{\partial \theta^{(l)}} J(\theta; x, y) = \frac{\partial J}{\partial f^{(l)}} \frac{\partial f^{(l)}_{\theta^{(l)}}}{\partial \theta^{(l)}} = \delta^{(l)} \frac{\partial f^{(l)}_{\theta^{(l)}}}{\partial \theta^{(l)}}$$

where $\frac{\partial f^{(l)}_{\theta^{(l)}}}{\partial \theta^{(l)}}$ is known

Backpropagation in General Cases

4. Sum gradients over all example to get overall gradient.

$$\nabla_{\theta^{(l)}} J(\theta) = \sum_{i=1}^{m} \nabla_{\theta^{(l)}} J\left(\theta; x^{(i)}, y^{(i)}\right)$$

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Convolution as Matrix Multiplication

$$\begin{bmatrix} 4 & 5 & 8 & 7 \\ 1 & 8 & 8 & 8 \\ 3 & 6 & 6 & 4 \\ 6 & 5 & 7 & 8 \end{bmatrix} * \begin{bmatrix} 1 & 4 & 1 \\ 1 & 4 & 3 \\ 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 122 & 148 \\ 126 & 134 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 5 & 8 & 7 & 1 & 8 & 8 & 8 & 3 & 6 & 6 & 4 & 6 & 5 & 7 & 8 \end{bmatrix}^T$$

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Convolution as Matrix Multiplication

4	5	8	7
1	8	8	8
3	6	6	4
6	5	7	8

*

1	4	1
1	4	3
3	3	1

⇒

122	148
126	134

1	4	1	0	1	4	3	0	3	3	1	0	0	0	0	0
	1	4	1	0	1	4	3	0	3	3	1	0			
				1	4	1	0	1	4	3	0	3	3	1	0
					1	4	1	0	1	4	3	0	3	3	1

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Convolution as Matrix Multiplication

1	4	1	0	1	4	3	0	3	3	1	0	0	0	0	0
	1	4	1	0	1	4	3	0	3	3	1	0			
				1	4	1	0	1	4	3	0	3	3	1	0
					1	4	1	0	1	4	3	0	3	3	1

*

4
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=

122
148
126
134

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Backpropagation in Convolution Layer

Forward propagation $x * w = y$

Backward propagation $\frac{\partial J}{\partial x} = flip(w) * \frac{\partial J}{\partial y}$

Gradient computation $x * \frac{\partial J}{\partial y} = \frac{\partial J}{\partial w}$

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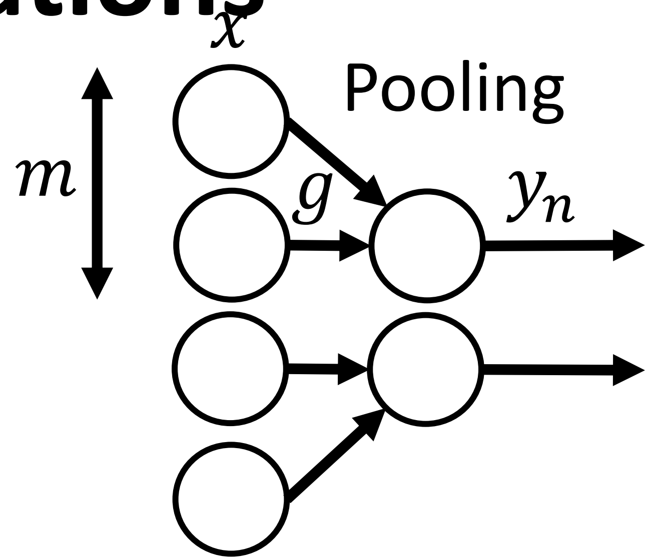
Pooling Layer: Subsampling Equations



Mean Pooling $g(x) = \frac{\sum_{k=1}^{m} x_k}{m}$ $\frac{\partial g}{\partial x} = \frac{1}{m}$

Max Pooling $\mathrm{g(x)} = \max(x)$ $\frac{\partial g}{\partial x_i} = \begin{cases} 1 & if\ x_i = \max(x) \\ 0 & otherwise \end{cases}$

L^p Pooling $g(x) = \|x\|_p = \left(\sum_{k=1}^{m} |x_k|^P\right)^{\frac{1}{P}}$ $\frac{\partial g}{\partial x_i} = \left(\sum_{k=1}^{m} |x_k|^P\right)^{\frac{1}{P}-1} . |x_i|^{p-1}$

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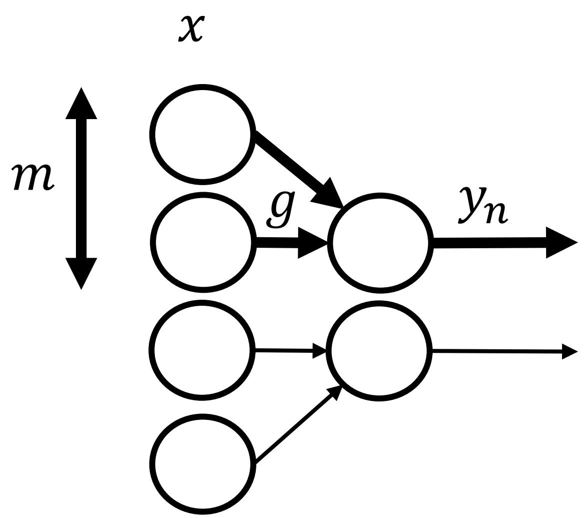
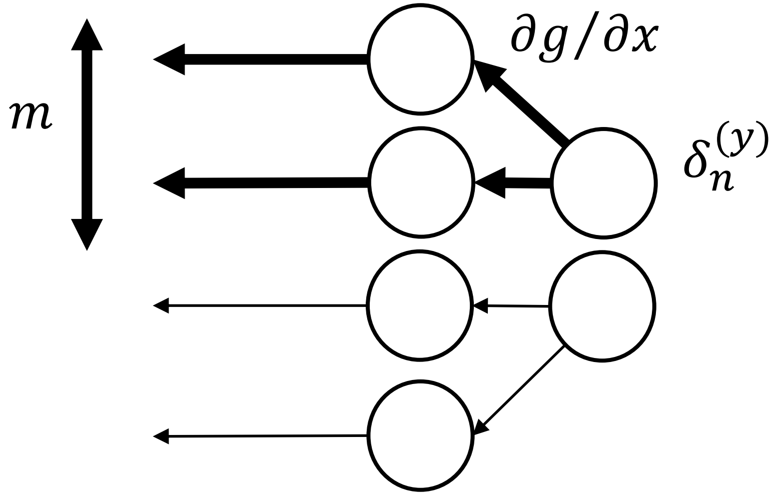
Backpropagation in the Pooling Layer



Forward propagation (subsampling)



Backward propagation (upsampling)